COMBINATORICA 1(2) (1981) 163—167

EQUIARBOREAL GRAPHS

C. D. GODSIL

Institut für Mathematik und Angewandte Geometrie
Montanuniversität Leoben, A—8700 Leoben, Austria

Received 20 September 1979

A graph X is said to be *equiarboreal* if the number of spanning trees containing a specified edge in X is independent of the choice of edge. We prove that any graph which is a colour class in an association scheme (and thus any distance regular graph) is equiarboreal. We note that a connected equiarboreal graph with M edges and n vertices has edge-connectivity at least $M/(n-1)$.

1. Introduction

P. Hell and E. Mendelsohn have (privately) asked the following question.

"Which graphs have the property that the number of spanning trees containing a given edge is independent of the edge?"

They call graphs with the property in question *equiarboreal*. Obviously any graph with a group of automorphisms acting transitively on its edges is equiarboreal. In this paper we establish a combinatorial condition which implies equiarboricity. We use this to prove that any graph which is a colour class in an association scheme is equiarboreal.

We also show that an equiarboreal graph with E edges and n vertices has edge connectivity at least $E/(n-1)$.

2. Definitions

X will be a graph on n vertices with adjacency matrix A. The diagonal matrix with i^{th} diagonal entry equal to the degree of the i^{th} vertex of X will be denoted by Δ. We define the polynomial $\tau(X, x)$ to be $\det(xI-(\Delta-A))$. From Theorem 7.5 of [1], for example, we know that $\tau(X, x)$ has zero constant term and that the absolute value of the coefficient of x in $\frac{1}{n}\tau(X, x)$ equals the number of spanning trees in X.

A *walk* in a graph is a sequence of vertices where consecutive vertices are adjacent. If i and j are vertices of X we use $W_{ij}(x)$ to denote the generating func-

AMS subject classification (1980): 05 C 05, 05 B 30; 15 A 18, 05 C 40 tion for the walks of length r from i to j ($r=0, 1, 2, \ldots$). If $v \in V(X)$ we use $\underset{\sim}{e}_v$ to denote the standard basis vector in $\mathbf{R}^n$ such that $A\underset{\sim}{e}_v$ is the column of A corresponding to v. It follows that

$$W_{ij}(x) = \sum_{n=0}^{\infty} \langle \underset{\sim}{e}_i, A^n \underset{\sim}{e}_j \rangle x^n,$$

where $A^0 = I$ and $\langle \cdot, \cdot \rangle$ represents the standard inner product in $\mathbf{R}^n$.

Let s be a non-negative integer. We call X *s-homogeneous* if, for any two vertices i and j in X at distance $t \leqq s$ from each other, $W_{ij}(x)$ depends only on t and not otherwise on the choice of i and j. We note that 0-homogeneous graphs have already been studied in [4] under the name "walk-regular" graphs.

We will consider an *association scheme* to be a commutative matrix algebra over the real numbers with a basis $A_0, A_1, \ldots, A_k$ of symmetric 0—1 matrices such that

$$A_0 = I \quad \text{and} \quad \sum_{i=0}^{k} A_i = J.$$

(Here, as elsewhere, J denotes the matrix with each entry equal to one.) Since A_i is a symmetric 0—1 matrix with zero diagonal we may view it as the adjacency matrix of a graph X_i ($i=1, 2, \ldots, k$). We say that such a graph is a *colour class* in the association scheme.

Association schemes are discussed briefly in [2]. Any distance-regular graph is a colour class in an association scheme. (For information on distance-regular graphs, see [1].)

3. Proposition. *If X is a colour class in an association scheme then it is* 1*-homogeneous.*

Proof. Suppose the matrices $A_0, A_1, \ldots, A_k$ are a basis for the association scheme and that A_1 is the adjacency matrix of X. Since the matrices A_i are a basis for an algebra, there are real numbers $\lambda_i(m)$ ($i=0, 1, \ldots, k$; $m=0, 1, \ldots$) such that

$$A_1^m = \sum_{i=0}^{k} \lambda_i(m) A_i.$$

Hence, if v and w are adjacent vertices in X,

$$\langle \underset{\sim}{e}_v, A_1^m \underset{\sim}{e}_v \rangle = \lambda_0(m) \langle \underset{\sim}{e}_v, A_0 \underset{\sim}{e}_v \rangle = \lambda_0(m)$$

and

$$\langle \underset{\sim}{e}_v, A_1^m \underset{\sim}{e}_w \rangle = \lambda_1(m) \langle \underset{\sim}{e}_v, A_1 \underset{\sim}{e}_w \rangle = \lambda_1(m).$$

Consequently $W_{vv}(x)$ is independent of the vertex v in X and $W_{vw}(x)$ is independent of the edge $\{v, w\}$ in X. Therefore X is 1-homogeneous. ∎

Let X and Y be graphs with adjacency matrices A and B respectively. We may define the categorical product of X and Y as the graph with adjacency matrix equal to $A \otimes B$. This product can be used to increase our supply of 1-homogeneous graphs.

4. Proposition. *The categorical product of* 1*-homogeneous graphs is* 1*-homogeneous.*

Proof. Let X, Y be 1-homogeneous graphs with adjacency matrices A and B respectively. Let Z be the categorical product of X and Y. Then Z has adjacency matrix $A\otimes B$ and if $u=(i, j)$, $v=(k, l)$ are vertices in Z,

$$\begin{aligned}\langle \underset{\sim}{e}_u, (A\otimes B)^r \underset{\sim}{e}_v\rangle &= \langle \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j, (A\otimes B)^r (\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l)\rangle \\ &= \langle \underset{\sim}{e}_i, A^r \underset{\sim}{e}_k\rangle \langle \underset{\sim}{e}_j, B^r \underset{\sim}{e}_l\rangle.\end{aligned}$$

Since u is adjacent to v in Z if and only if both i is adjacent to k in X and j is adjacent to l in Y, it follows that Z is 1-homogeneous if X and Y are. ∎

By Propositions 3 and 4, the categorical product of any two distance regular graphs is 1-homogeneous. In general such a product will not be distance regular, although it will be a colour class in an association scheme. We know of no 1-homogeneous graph which is not a colour class in an association scheme, but this does not tempt us to conjecture that there are none.

We come now to our main result.

5. Theorem. *A* 1*-homogeneous graph is equiarboreal.*

This theorem is a consequence of the following proposition.

6. Proposition. *Let X be a regular graph of degree d. Let $e=\{1, 2\}$ be an edge in X and let $X\setminus e$ denote the graph obtained when e (but not its end-vertices) is removed from X. Then*

$$\frac{\tau(X\setminus e, x)}{\tau(X, x)} = 1-y\left(W_{11}(y)+W_{22}(y)-2W_{12}(y)\right),$$

where $y=1/(d-x)$.

Proof. Let $\underset{\sim}{u}^t$ be the vector $[1, -1, 0, \ldots, 0]$ in $\mathbf{R}^n$, where $n=|V(X)|$. Let E be the $n\times n$ matrix $\underset{\sim}{u}\underset{\sim}{u}^t$. Then, setting $B=xI+A-\Delta$ and abbreviating $\tau(X\setminus e, x)$ and $\tau(X, x)$ to $\tau(X\setminus e)$ and $\tau(X)$ respectively, we have $\tau(X\setminus e)=\det(B+E)$. This determinant can be written as sum of the determinants of the 2^n matrices obtained by replacing in turn each subset of the columns of B by the corresponding subset of columns of E. Since E has rank 1 and since $n-2$ of its columns are empty we get

$$\tau(X\setminus e) = \det B+(\det B_1+\det B_2),$$

where B_i is obtained by replacing the i^{th} column of B with the i^{th} column of E for $i=1, 2$. If we let B_{ij} denote the cofactor of the ij-entry of B then

$$\begin{aligned}\det B_1 &= B_{11}+B_{12}\\ \det B_2 &= B_{21}+B_{22}.\end{aligned}$$

Hence $\tau(X\setminus e)=\tau(X)+(B_{11}+B_{12}+B_{21}+B_{22})$. Now we also have

$$\underset{\sim}{u}^t B^* \underset{\sim}{u} = B_{11}+B_{12}+B_{21}+B_{22},$$

where B^* is the adjoint of B. Thus

$$\tau(X\setminus e) = \tau(X)+\underset{\sim}{u}^t B^* \underset{\sim}{u}.$$

Since $\tau(X)=\det B$, this yields

$$\frac{\tau(X\setminus e)}{\tau(X)} = 1+\underset{\sim}{u}^t B^{-1}\underset{\sim}{u} = 1+\underset{\sim}{u}^t(xI+A-\Delta)^{-1}\underset{\sim}{u}.$$

As X is regular with degree d, $\Delta=dI$. Setting $y=1/(d-x)$ we find that

$$\frac{\tau(X\setminus e)}{\tau(X)} = 1-y\underset{\sim}{u}^t(I-yA)^{-1}\underset{\sim}{u}.$$

Finally we have

$$(I-yA)^{-1} = \sum_{i=0}^{\infty} y^r A^r$$

and so the ij-entry of $(I-yA)^{-1}$ is just $W_{ij}(y)$. Hence,

$$\frac{\tau(X\setminus e)}{\tau(X)} = 1-y\big(W_{11}(y)+W_{22}(y)-W_{12}(y)-W_{21}(y)\big).$$

Since X is a graph, $W_{12}(y)=W_{21}(y)$, and so the proof is complete. ∎

The preceding proposition shows that if X is 1-homogeneous then $\tau(X\setminus e, x)$ is independent of the edge e. By our introductory remarks on $\tau(X, x)$, it follows that the number of spanning trees in X which do not use e is independent of the choice of e. Consequently X is equiarboreal. Thus we have established Theorem 5.

We finish by noting an interesting property of equiarboreal graphs.

7. Proposition. *If X is a connected equiarboreal graph on n vertices with M edges then the edge-connectivity is at least $M/(n-1)$.*

Proof. Assume X has T spanning trees and that each edge in X lies on exactly t spanning trees. The number of pairs (S, e), where S is a spanning tree in X and e is an edge in S, is $tM=T(n-1)$ and so $T/t=M/(n-1)$.

Let $\mathscr{C}$ be an edge cut set of X. Each spanning tree of X contains at least one edge in $\mathscr{C}$. Hence we have

$$T \leqq t|\mathscr{C}|.$$

Consequently $|\mathscr{C}|\geqq M/(n-1)$, as required. ∎

It follows that a connected equiarboreal graph which is regular of degree d has edge-connectivity strictly greater than $d/2$. Hence the edge connectivity of distance regular graphs increases with their degree. This does not appear to have been known previously.

It is known that the edge-connectivity of a graph equals its degree when it is vertex-transitive (see [5 (Problem 12.14)], [6]) or strongly regular (this follows from Theorem 6 of [8], which asserts that the edge-connectivity of a graph with diameter two equals its minimum degree). These results motivate the following:

8. Conjecture. *Let X be a graph which is a colour class in an association scheme. Then its edge-connectivity equals its degree.*

The problem of determining the vertex-connectivity of these graphs also remains open. (Of course, by the results of [7, 9] we know that a vertex transitive graph with degree d has vertex-connectivity at least $2(d+1)/3$). M. Fiedler has shown in [3] that a regular graph with degree d and second largest eigenvalue α has vertex-connectivity at least $d-\alpha$. For line graphs of Steiner triple systems this gives a bound which is approximately $\frac{2}{3}d$. The writer has an elementary argument which shows that the vertex-connectivity of a strongly regular graph is at least $\frac{1}{3}(d+1)$, but this seems unreasonably low.

Acknowledgement. This work arose from the author's participation in the 1979 Summer Research Workshop in Algebraic Combinatorics of the Canadian Math. Society, which was held at Simon Fraser University, Burnaby B. C.

References

[1] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press (1974).
[2] P. J. Cameron and J. H. van Lint, *Graph Theory, Coding Theory and Block Designs*, L.M.S. Lecture Notes 19, Cambridge University Press (1975).
[3] M. Fiedler, Algebraic connectivity of graphs, *Czech. Math. J.* **23** (1973), 298—305.
[4] C. D. Godsil and B. D. McKay, Feasibility conditions for the existence of walk-regular graphs, *Lin. Alg. Appl.* **30** (1980), 51—61.
[5] L. Lovász, *Combinatorial Problems and Exercises*, Akadémiai Kiadó—North-Holland (1979).
[6] B. D. McKay, private communication.
[7] W. Mader, Über den Zusammenhang symmetrischer Graphen, *Arch. Math.* **21** (1970), 331—336.
[8] J. Plesník, Critical graphs of given diameter, *Acta Fac. Rerum Natur. Univ. Comenianae, Math.* **30** (1975), 71—93.
[9] M. E. Watkins, Connectivity of transitive graphs, *J. Combinatorial Theory* **8** (1970), 23—29.